International Journal of Applied Mathematics
& Statistical Sciences (IJAMSS)
ISSN(P): 2319-3972; ISSN(E): 2319-3980
Vol. 5, Issue 5, Aug - Sep 2016; 1-8

International Academy of Science, Engineering and Technology
Connecting Researchers; Nurturing Innovations

STABILITY ANALYSIS OF A DELAYED SIR MODEL WITH NONLINEAR INCIDENCE RATE

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ABSTRACT

In this paper a delayed SIR model with exponential demographic structure and the nonlinear incidence rate is formulated. We show if the basic reproductive number, denoted, R_0, is less than unity, disease free equilibrium is stable. Moreover, we prove that $R_0 > 1$, the endemic equilibrium is locally stable without delay and the endemic equilibrium is stable if the delay is under some condition. Finally a numerical example is also included to illustrate the effectiveness of the proposed model.

KEYWORDS: SIR Epidemic Model, the Basic Reproduction Number, Stability, Time Delay, Hurwitz Criterion, Hopf Bifurcation

AMS Subject Classifications: 34C07, 34D23, 93A30, 93D20

1. INTRODUCTION

In recent years, more and more delayed models have been investigated during the study of epidemic models [3, 5, 6, 8, 10, 11, 13, 14]. Hethcote and Van den Driessche [4] have considered an SIS epidemic model with constant time delay which accounts for duration of infectiousness. Beretta et al. [1] have studied global stability in an SIR epidemic model with distributed delay that describes the time which it takes for an individual to lose infectiousness. Kaddar et al. [7] considered a delayed SIR model with a saturated incidence rate $\frac{\beta S(t)I(t)}{1+\alpha_1 S(t)+\alpha_2 I(t)}$ as follows:

$$\frac{dS}{dt} = A - \mu S(t) - \frac{\beta S(t)I(t)}{1+\alpha_1 S(t)+\alpha_2 I(t)},$$
$$\frac{dI}{dt} = \frac{\beta S(t-\tau)I(t-\tau)}{1+\alpha_1 S(t-\tau)+\alpha_2 I(t-\tau)} - (\mu+\alpha+\gamma)I(t),$$
$$\frac{dR}{dt} = \gamma I(t) - \mu R(t).$$

The characteristic of this model is: the saturated incidence rate $\frac{\beta S(t)I(t)}{1+\alpha_1 S(t)+\alpha_2 I(t)}$ which includes the three forms: $\beta S(t)I(t)$ (if $\alpha_1=\alpha_2=0$), $\frac{\beta S(t)I(t)}{1+\alpha_1 S(t)}$ (if $\alpha_2=0$) and $\frac{\beta S(t)I(t)}{1+\alpha_2 I(t)}$ (if $\alpha_1=0$) saturated with the susceptible

$S(t)$ and infective $I(t)$ individuals. The inclusion of time delay into susceptible $S(t)$ and infective $I(t)$ individuals in incidence rate, only on the second equation, because susceptible individuals infected at time $t-\tau$ is able to spread the disease at time t. Rihan et al. [9] considered a qualitative analysis of delayed SIR epidemic model with saturated incidence rate $\frac{\beta S(t)I(t)}{1+\sigma S(t)}$ as

$$\frac{dS}{dt} = rS(t)\left(1-\frac{S(t)}{K}\right) - \frac{\beta S(t)I(t-\tau)}{1+\sigma S(t)},$$
$$\frac{dI}{dt} = \frac{\beta S(t)I(t-\tau)}{1+\sigma S(t)} - aI(t) - \alpha I(t),$$
$$\frac{dR}{dt} = \alpha I(t) - \delta R(t).$$

Here parameters r is the logistic growth rate, K is carrying capacity, σ is the saturation factor that measures the inhibitory effect, $\frac{1}{\tau}$ is the incubation period [2,12], α is the recovery rate, δ is the natural death rate due to causes unrealed to infection and a is the infected hosts die rate which includes both the natural death rate plus the disease induced death rate.

In this paper we consider a delayed SIR model with the nonliner incidence rate $\frac{\beta S(t-\tau)I(t-\tau)e^{-\mu\tau}}{1+\alpha S(t-\tau)}$. We also analyze the stability and the existence of Hopf bifurcation.

2. DELAYED SIR EPIDEMIC MODEL

In this section, we consider the following SIR model with the non linear incidence rate $\frac{\beta S(t-\tau)I(t-\tau)e^{-\mu\tau}}{1+\alpha S(t-\tau)}$. Let $S(t)$ is the number of susceptible individual, $I(t)$ is the number of infective individual, and $R(t)$ is the number of recovered individuals, then we have the following model

$$\frac{dS}{dt} = rS(t)\left(1-\frac{S(t)}{K}\right) - \frac{\beta S(t-\tau)I(t-\tau)e^{-\mu\tau}}{1+\alpha S(t-\tau)} - \mu S(t),$$
$$\frac{dI}{dt} = \frac{\beta S(t-\tau)I(t-\tau)e^{-\mu\tau}}{1+\alpha S(t-\tau)} - (\mu+\rho+\delta)I(t),$$
$$\frac{dR}{dt} = \rho I(t) - \mu R(t). \tag{2.1}$$

The parameter r is the logistic growth rate, K is the carrying capacity, $\mu > 0$ is the rate of natural death such that $r > \mu$, $\delta > 0$ is the rate of disease related death, $\rho > 0$ is the rate of recovery, $\frac{1}{\tau}$ is the incubation period and α is the parameters that measures infections with the inhibitory effect.

The first two equation in system (2.1) do not depend on the third equation, and therefore this equation can be omitted without lose of generality. Hence system (2.1) can be rewritten as

$$\begin{aligned}\frac{dS}{dt} &= rS(t)\left(1-\frac{S(t)}{K}\right)-\frac{\beta S(t-\tau)I(t-\tau)e^{-\mu\tau}}{1+\alpha S(t-\tau)}-\mu S(t),\\ \frac{dI}{dt} &= \frac{\beta S(t-\tau)I(t-\tau)e^{-\mu\tau}}{1+\alpha S(t-\tau)}-(\mu+\rho+\delta)I(t).\end{aligned} \tag{2.2}$$

Propostion: For the model system (2.2), there always exit infection free equilibrium $E_0=(0,0),\ E_1=\left(\frac{(r-\mu)K}{r},0\right)$. If

$$R_0=\frac{K(r-\mu)[\beta e^{-\mu\tau}-\alpha(\mu+\rho+\delta)]}{r(\mu+\rho+\delta)} \tag{2.3}$$

There also exits an endemic equilibrium $E^*=(S^*,I^*)$, where

$$S^*=\frac{(\mu+\rho+\delta)}{\beta e^{-\mu\tau}-\alpha(\mu+\rho+\delta)},\mathrm{I}^*=\frac{rS^{*2}}{K(\mu+\rho+\delta)}[R_0\text{-}1].$$

3. LINEAR STABILITY ANALYSIS

The characteristic equation for the model (2.2), is given by

$$\begin{vmatrix}(r-\mu)-\frac{2rS}{K}-\frac{\beta Ie^{-(\mu+\lambda)\tau}}{(1+\alpha S)^2}-\lambda & -\frac{\beta Se^{-(\mu+\lambda)\tau}}{1+\alpha S}\\ \frac{\beta Ie^{-(\mu+\lambda)\tau}}{(1+\alpha S)^2} & \frac{\beta Se^{-(\mu+\lambda)\tau}}{1+\alpha S}-(\mu+\rho+\delta)-\lambda\end{vmatrix}=0 \tag{3.1}$$

Theorem 3.1: E_0 is always a saddle point and there can not be total extinction of the system (2.2).

Proof Using (3.1), the characteristic equation at $E_0=(0,0)$ reduces to

$$[\lambda-(r-\mu)][\lambda+(\mu+\rho+\delta)]=0 \tag{3.2}$$

Obviously (3.2) has a positive root $\lambda=r-\mu.$ Thus E_0 is always unstable (saddle point).

Theorem 3.2: The infection free equilibrium $E_1=\left(\frac{K(r-\mu)}{r},0\right)$ is asymptotically stable when $R_0<1$, and unstable when $R_0>1$, and linearly neutrally stable if $R_0=1$.

Proof. Using (3.1), the characteristic equation at $E_1 = \left(\frac{K(r-\mu)}{r}, 0\right)$ is

$$\begin{vmatrix} (r-\mu)-\lambda & -\dfrac{\beta K(r-\mu)e^{-(\mu+\lambda)\tau}}{r\left[1+\alpha\dfrac{K(r-\mu)}{r}\right]} \\ 0 & \dfrac{\beta K(r-\mu)e^{-(\mu+\lambda)\tau}}{r\left[1+\alpha\dfrac{K(r-\mu)}{r}\right]} -(\mu+\rho+\delta)-\lambda \end{vmatrix} = 0$$

This implies

$$[\lambda+(r-\mu)]\left[\lambda+(\mu+\rho+\delta)\left\{1-\frac{rR_0+\alpha(r-\mu)Ke^{-\lambda\tau}}{r+\alpha K(r-\mu)}\right\}\right]=0. \tag{3.3}$$

The two roots of (3.3) are real and negative if $R_0 < 1$ (when $\tau = 0$). The equilibrium E_1 is asymptotical stable. When $\tau > 0$, we suppose (3.3) has a purely imaginary root $\lambda = \omega i$, then separating real and imaginary parts, we have

$$\begin{aligned} -\omega^2+(r-\mu)(\mu+\rho+\delta) &= (r-\mu)(\mu+\rho+\delta)\frac{rR_0+\alpha(r-\mu)K}{r+\alpha(r-\mu)K}\cos\omega\tau \\ &\quad +\omega(\mu+\rho+\delta)\frac{rR_0+\alpha(r-\mu)K}{r+\alpha(r-\mu)K}\sin\omega\tau \end{aligned} \tag{3.4}$$

$$\begin{aligned} \left[(\mu+\rho+\delta)+(r-\mu)\right]\omega &= \omega(\mu+\rho+\delta)\frac{rR_0+\alpha(r-\mu)K}{r+\alpha(r-\mu)K}\cos\omega\tau \\ &\quad -(r-\mu)(\mu+\rho+\delta)\frac{rR_0+\alpha(r-\mu)K}{r+\alpha(r-\mu)K}\sin\omega\tau \end{aligned} \tag{3.5}$$

Hence

$$\omega^2 = (\mu+\rho+\delta)^2\left[\left\{\frac{rR_0+\alpha(r-\mu)K}{r+\alpha(r-\mu)K}\right\}^2 - 1\right] \tag{3.6}$$

When $R_0 < 1$, then there are no positive real roots ω.

This is complete the proof.

Theorem 3.3: (i) The endemic equilibrium E^* is asymptotically stable if $1 < R_0 < 2 + \frac{1}{\alpha S^*}$, when $\tau = 0$.

(ii) When $\tau > 0$, $R_0 > 1$ suppose $\frac{1}{1+\alpha S^*}\left(1-\frac{1}{R_0}\right)\left[\frac{1}{1+\alpha S^*}\left(1-\frac{1}{R_0}\right)-\frac{2(\mu+\rho+\delta)}{r-\mu}\right] < \left(1-\frac{2}{R_0}\right)^2$, then there exit $\tau_1 > 0$ such that $\tau \in [0, \tau_1]$ the endemic equilibrium E^* is stable, and unstable when $\tau > \tau_1$.

Proof The characteristic equation for the system (2.2) at $E^* = (S^*, I^*)$ is given by

$$\begin{vmatrix} (r-\mu) - \frac{2rS^*}{K} - \frac{\beta I^* e^{-(\mu+\lambda)\tau}}{(1+\alpha S^*)^2} - \lambda & -\frac{\beta S^* e^{-(\mu+\lambda)\tau}}{1+\alpha S^*} \\ \frac{\beta I^* e^{-(\mu+\lambda)\tau}}{(1+\alpha S^*)^2} & \frac{\beta S^* e^{-(\mu+\lambda)\tau}}{1+\alpha S^*} - (\mu+\rho+\delta) - \lambda \end{vmatrix} = 0.$$

This implies

$$\begin{vmatrix} (r-\mu) - \frac{2(r-\mu)}{R_0} - \frac{r-\mu}{1+\alpha S^*}\left(1-\frac{1}{R_0}\right)e^{-\lambda\tau} - \lambda & -(\mu+\rho+\delta)e^{-\lambda\tau} \\ \frac{r-\mu}{1+\alpha S^*}\left(1-\frac{1}{R_0}\right)e^{-\lambda\tau} & (\mu+\rho+\delta)\left(e^{-\lambda\tau}-1\right) - \lambda \end{vmatrix} = 0.$$

$$\begin{aligned} &\lambda^2 + \left[-(r-\mu)\left(1-\frac{2}{R_0}\right) + (\mu+\rho+\delta)\left(1-e^{-\lambda\tau}\right) + \frac{r-\mu}{1+\alpha S^*}\left(1-\frac{1}{R_0}\right)e^{-\lambda\tau}\right]\lambda \\ &+(\mu+\rho+\delta)\left[-(r-\mu)\left(1-\frac{2}{R_0}\right)\left(1-e^{-\lambda\tau}\right) + \frac{r-\mu}{1+\alpha S^*}\left(1-\frac{1}{R_0}\right)e^{-\lambda\tau}\right]. \end{aligned} \tag{3.7}$$

Introducing

$$Q_1 = (r-\mu)\left(1-\frac{2}{R_0}\right), \qquad Q_2 = \frac{r-\mu}{1+\alpha S^*}\left(1-\frac{1}{R_0}\right), \qquad Q_3 = (\mu+\rho+\delta). \tag{3.8}$$

Then the characteristic equation (3.7) can be rewritten in the form

$$\lambda^2 + p_1\lambda + p_2 + [q_1\lambda + q_2]e^{-\lambda\tau} = 0. \tag{3.9}$$

Where

$$p_1 = -Q_1 + Q_3 = -(r-\mu)\left(1-\frac{2}{R_0}\right) + (\mu+\rho+\delta),$$

$$p_2 = -Q_1Q_3 = -(r-\mu)(\mu+\rho+\delta)\left(1-\frac{2}{R_0}\right),$$

$$q_1 = -Q_3 + Q_2 = -(\mu+\rho+\delta) + \frac{(r-\mu)}{1+\alpha S^*}\left(1-\frac{1}{R_0}\right),$$

$$q_2 = Q_1Q_3 + Q_2Q_3 = \left[1-\frac{2}{R_0}+\frac{1}{1+\alpha S^*}\left(1-\frac{1}{R_0}\right)\right](r-\mu)(\mu+\rho+\delta).$$

When $\tau = 0$, the characteristic equation (3.9) becomes $\lambda^2 + a_1\lambda + a_2 = 0$. Where $a_1 = p_1 + q_1 = (r-\mu)\left[-1+\frac{2}{R_0}+\frac{1}{1+\alpha S^*}\left(1-\frac{1}{R_0}\right)\right]$, $a_2 = p_2 + q_2 = Q_2Q_3 > 0$. Therefore, if $a_1 > 0$ i.e. $R_0 < 2+\frac{1}{\alpha S^*}$, $a_2 > 0$ (when $R_0 > 1$), then by the Hurwitz criterion, we can know the endemic equilibrium E^* is stable.

When $\tau > 0$, we suppose equation (3.9) has a purely imaginary root $\lambda = \omega i$, the separating real and imaginary parts.

$$\omega^2 - p_2 = q_2 \cos\omega\tau + q_1\omega \sin\omega\tau, \tag{3.10}$$

$$p_1\omega = q_2 \sin\omega\tau - q_1\omega\cos\omega\tau. \tag{3.11}$$

Hence

$$\omega^4 + a_3\omega^2 + a_4 = 0 \tag{3.12}$$

Where

$$a_3 = p_1^{\,2} - 2p_2 - q_1^{\,2}$$
$$= (r-\mu)^2\left(1-\frac{2}{R_0}\right)^2 - \frac{(r-\mu)^2}{1+\alpha S^*}\left(1-\frac{1}{R_0}\right)\left[\frac{1}{1+\alpha S^*}\left(1-\frac{1}{R_0}\right)-\frac{2}{r-\mu}(\mu+\rho+\delta)\right],$$
$$a_2 = p_2^{\,2} - q_2^{\,2} < 0$$

If $a_3 > 0$, i.e. $\frac{1}{1+\alpha S^*}\left(1-\frac{1}{R_0}\right)\left[\frac{1}{1+\alpha S^*}\left(1-\frac{1}{R_0}\right)-\frac{2(\mu+\rho+\delta)}{r-\mu}\right] < \left(1-\frac{2}{R_0}\right)^2$, then equation (3.12) has at least one positive root, say $\omega_1 > 0$.

Now, we turn to the bifurcation analysis. We use the delay τ as bifurcation parameter, let $\lambda(\tau) = \gamma(\tau) + i\omega(\tau)$ be the eigenvalue of equation (3.9) such that for some initial value of the bifurcation parameter τ_1, we have $\gamma(\tau_1) = 0$ and $\omega(\tau_1) = \omega_1$. From (3.10) and (3.11) we have

$$\tau_1 = \frac{1}{\omega_1}\arccos\left[\frac{(q_2 - p_1q_1)\omega_1^{\,2} - p_2q_2}{q_1^{\,2}\omega_1^{\,2} + q_2^{\,2}}\right] + \frac{2j\pi}{\omega_1},\ j = 0, 1, 2, .. \tag{3.13}$$

Differentiate w.r.t. τ_1, we get

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{2\lambda + p_1}{-\lambda(\lambda^2 + p_1\lambda + p_2)} + \frac{q_1}{q_2\lambda} - \frac{q_1^{\,2}}{q_2(q_1\lambda + q_2)} - \frac{\tau}{\lambda}. \tag{3.14}$$

Thus

$$\text{sign}\left\{\text{Re}\frac{d\lambda}{d\tau}\right\}_{\lambda=\omega i} = \text{sign}\left\{\text{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}\right\}_{\lambda=\omega i} = \text{sign}\left[\frac{P}{Q}\right],$$

Where

$P = q_1^2\omega_1^4 + \omega_1^2(2q_2^2) + q_2^2(p_1^2 - 2p_2) + p_2^2q_1^2, Q = \left[p_1^2\omega_1^2 + (\omega_1^2 - p_2)^2\right]\left[q_2^2 + q_1^2\omega_1^2\right]$ which are positive when $R_0 > 1$.

Thus we have $\left\{\text{Re}\frac{d\lambda}{d\tau}\right\}_{\tau=\tau_1} > 0,$ by continuity the real part of $\lambda(\tau)$ becomes positive when $\tau > \tau_1$ and the steady state becomes unstable. A Hopf bifurcation occurs when τ passes through the critical value τ_1.

4. EXAMPLE

In this section, we present some numerical results of system (2.2) at different τ of supporting the theoretical analysis in section 3

We take the parameters for endemic equilibrium without delay as follows: $S^0 = 25, I^0 = 4, R^0 = 6, r = .4, \beta = 0.2, \alpha = 0.09, \mu = 0.3, \rho = 0.3, \delta = 0.2, \tau = 0, K = 60.$ We have $R_0 = 2.4 > 1, S^* = 6.25, I^* = 0.4557,$ then by theorem 3.3 (i), $S(t)$ and $I(t)$ approach to their steady-state values without delay, the disease will be exist.

We give the parameters for endemic equilibrium with time delay in system (2.2) as follows: $S^0 = 25, I^0 = 4, R^0 = 6, r = .4, \beta = 0.2, \alpha = 0.09, \mu = 0.3, \rho = 0.3, \delta = 0.2, \tau = 1, K = 50.$

Then we get $R_0 = 1.136 > 1, S^* = 11, I^* = 1.21,$ by theorem 3.3, $S(t)$ and $I(t)$ approach to their steady-state values with delay, the disease will be exist.

5. CONCLUSIONS

We have analytically studied a delayed SIR model with the exponential demographic structure and the nonlinear incidence. We found the sufficient condition of the stability for the endemic and disease free equilibrium of the model. When $R_0 \le 1,$ the disease free steady state is stable, no other equilibria exist. When $R_0 > 1,$ a unique endemic equilibrium exists and stable under some condition with and without delay.

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